

First and Second Laws of Thermodynamics in Modified Hořava-Lifshitz $F(R)$ gravity

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Abstract

In this paper we discuss the thermodynamics of the apparent horizon in $F(R)$ Hořava-Lifshitz gravity in equilibrium and non-equilibrium ensembles. We show that the second law of thermodynamics can be satisfied in this non relativistic theory.

Keywords: $F(R)$ Hořava-Lifshitz gravity, Equilibrium and non-equilibrium thermodynamic;, First and second law of thermodynamics.

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1 Introduction

Different types of the observational data, namely type Ia supernovae , cosmic microwave background (CMB) , large scale structure , baryon acoustic oscillations , and weak lensing show that our Universe is accelerating [1, 2] . Modified gravity is the simplest way to address this accelerating behavior. In this approach one simply modifies the original Einstein-Hilbert action by an arbitrary function of the curvature term(s) like $R, R_{\mu\nu}, R^\alpha_{\beta\mu\nu}, \dots$. Such types of modifications originally proposed in [3] and recently revisited in light of the current acceleration of the Universe [4],[5]. The simplest model of modified gravity is a class of models, alled $F(R)$ gravity in which one replaces the classical Hilbert-Einstein action of gravity by an arbitrary function of R , the Ricci scalar term [7]:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R) \quad (1)$$

Different aspects of this type of gravity studied in literature [8]. One significant result is that the $F(R)$ gravity is Lorentz invariant like Einstein gravity because of its invariant form under global coordinate transformations. This feature is a basic concept and is valid even in the teleparallel gravity [9]. Modified gravity looks very similar to the Einstein gravity in solar system when all solar tests by a highly precision are done, we observe that small deviations from this theory can satisfy these local tests. This is one of the most important advantages of $F(R)$ gravity [10, 11] . Based on this fact, we are able to successfully reconstruct viable models of $f(R)$ gravity for cosmological applications [12, 13, 14, 15]. It was proven that the $F(R)$ models are responsible for accelerating expansion as well as to describe the dark matter problem in the rotation curves of different galaxies without the need for dark matter . This issue is vastly studied by authors [17, 18, 19, 20, 21](see for example [4, 22].) There are other types of modified gravity theories for example when the Ricci scalar R is coupled to the matter Lagrangian density L_m both in metric approach [23], [24], [25, 26] and in the Palatini formulation [27]. Also another version of these non-minimally coupled models has been proposed in [28], where they are assuming that the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the matter Lagrangian L_m in the form of $f(R, L_m)$ gravity, this model originally was proposed in [29].

Non relativistic regimes of gravity is also important for example to improve the propagator of graviton in ultraviolet regime. In (2009) a new approach to the quantum gravity proposed by Hořava based on idea of Lifshitz in quantum systems [30, 31, 32, 33]. The proposal is to take into account different space-time footing. On account of this assumption, the Lorentz symmetry is broken consequently theory renormalizable at quantum level. We'll review this proposal more technically in Sec. (2). The theory called as Hořava-Lifshitz theory and widely studied in literature [34]-[47]. Hořava-Lifshitz cosmology is also widely studied by authors [48, 49, 50, 51, 52]. In particular, one can examine specific solution subclasses [53, 54, 55], the perturbation spectrum [56, 57, 58, 59, 60,

61, 62], the gravitational wave production [63, 64], the matter bounce [65, 66, 67], the black hole properties [68, 69, 70, 71, 72, 73, 74], the cosmic string solutions [75] the dark energy phenomenology [76, 77, 78, 79], the astrophysical phenomenology [80, 81], exact solutions to field equations [82]-[85], quantum spectrum of black holes [86] and etc [38, 39, 40].

Notivated by $F(R)$ gravity a fully foliation-preserving diffeomorphisms invariance version of HL theory proposed in [87],[88]. This viable extension of Hořava-Lifshitz theory has the following remarkable results:

- It was demonstrated that cosmological equation on the spatially-flat sector of space time are consistent with the constraint equations.
- Due to the existence of the de Sitter solutions in several versions of theory, it is possible to consistently unify the early-time inflation with the late-time acceleration.
- It reduces to the classical cosmological equations with a special choice of parameters.
- The cosmological equations do coincide with the ones for the related, convenient $F(R)$ gravity. This means the cosmological history of Hořava-Lifshitz $F(R)$ gravity will be just the same as for its convenient version. For the general version of the theory the situation turns out to be more complicated.

The cosmological viability of the model was demonstrated in $F(R)$ -Hořava-Lifshitz theory but it is not east to construct an appropriate consistent Hamiltonian formalism [89],[90]. $F(R)$ -HL theory has many possible uses in the cosmology and has also been investigated as a potential valid modification of the original HL theory [91]-[103]. However, although the cosmological effects of the $F(R)$ Hořava-Lifshitz on the physical properties of Universe was demonstrated over last years ago, little attention has been paid to the thermodynamics of an appropriate $F(R)$ Hořava-Lifshitz model. The present paper presents a set of criteria for investigating thermodynamic laws. On the basis of these criteria it then describes the validation of a first and second laws using apparent and event horizons. This combination of two basically distinct laws formed a novel interpretation in which the incorporation of $F(R)$ part significantly increased viability.

This paper is organized as follows: In Sec. (2) we present a brief review to the original HL theory. In Sec. (3) we provide the basicl Eqs. in $F(R)$ Hořava-Lifshitz theory. In Sec. (4) we study cosmological solutions. Sec. (5) is devoted to study equilibrium and non-equilibrium thermodynamics. In Sec. (6) we study equilibrium regime. We summarize in final section.

2 Review of Hořava-Lifshitz gravity with detailed condition

Hořava-Lifshitz theory is a power-counting renormalizable, ultraviolet complete theory of gravity [31, 32, 30, 33]. The fixed point of theory in infrared regime is Einstein gravity. In the UV regime, HL theory has a fixed point with an anisotropic, Lifshitz scaling between time and space of the form $x^i \rightarrow \ell x^i$, $t \rightarrow \ell^z t$, where ℓ , z , x^i and t are the scaling factor, dynamical critical exponent, spatial coordination and temporal coordination, respectively. Let us to start by decomposing metric in ADM formalism. Following from the Arnowitt-Deser-Misner formalism (ADM) decomposition of the metric [104].[105],[106], and the Einstein equations, the dynamical fields are the fields $N(t, x)$, $N_i(t, x)$, $g_{ij}(t, x)$ corresponding to the *lapse*, *shift* and *spatial metric*. The general metric in the so-called ADM decomposition in a 3 + 1 spacetime [104]-[106] is

$$ds^2 = -N^2 dt^2 + g_{ij}^{(3)}(dx^i + N^i dt)(dx^j + N^j dt), \quad (2)$$

where $i, j = 1, 2, 3$, N is the so-called Lapse variable and N^i is the shift 3-vector. The Ricci scalar in the general relativity (*GR*) can be written in terms of the metric and we have

$$R = K_{ij}K^{ij} - K^2 + R^{(3)} + 2\nabla_\mu(n^\mu\nabla_\nu n^\nu - n^\nu\nabla_\nu n^\mu), \quad (3)$$

here we define the extrinsic curvature $K = g^{ij}K_{ij}$, K_{ij} , the spatial scalar curvature of g_{ij} is denoted by $R^{(3)}$ and n^ν is a unit normal vector in con dimension one time sliced metric $t = 0$. We can define the extrinsic curvature as

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij}^{(3)} - \nabla_i^{(3)}N_j - \nabla_j^{(3)}N_i), \quad (4)$$

the lapse variable N in the original model is taken to be just time-dependent and the condition of projectability is hold. Using the foliation-preserving diffeomorphisms invariance, we can fix $N = 1$. This version of theory is called projectable and it was demonstrated that may cause problems with Newton's law in weak limit [109]. To preserve Newtonian gravity in weak regimes, we need to work in the framework of the non-projectable $F(R)$ -model [?]. For the non-projectable case, the Newton's law could be restored by the "healthy" extension of the original Hořava gravity of [109].

The action of Hořava-Lifshitz theory for $z = 3$ is

$$S = \int_M dt d^3x \sqrt{g} N (\mathcal{L}_K - \mathcal{L}_V) \quad (5)$$

here the space-covariant derivative on a covector v_i is defined by $\nabla_i v_j \equiv \partial_i v_j - \Gamma_{ij}^l v_l$ where Γ_{ij}^l are the spatial Christoffel symbols, by g we mean the determinant of the 3-metric g_{ij} and $N = N(t)$ is a dimensionless homogeneous gauge field. The kinetic term is

$$\mathcal{L}_K = \frac{2}{\kappa^2} \mathcal{O}_K = \frac{2}{\kappa^2} (K_{ij}K^{ij} - \lambda K^2)$$

Here N_i is a gauge field with scaling dimension $[N_i] = z - 1$.

The *potential* term \mathcal{L}_V of the $(3 + 1)$ -dimensional theory is determined by the *principle of detailed balance* is given by the following expression:

$$\mathcal{L}_V = \alpha_6 C_{ij} C^{ij} - \alpha_5 \epsilon_l^{ij} R_{im} \nabla_j R^{ml} + \alpha_4 [R_{ij} R^{ij} - \frac{4\lambda - 1}{4(3\lambda - 1)} R^2] + \alpha_2 (R - 3\Lambda_W) \quad (6)$$

The coupling constants α_i define by

$$\alpha_2 = \frac{\alpha_4 \Lambda_w}{3\lambda - 1}, \quad \alpha_4 = \frac{\kappa^2 \mu^2}{8}, \quad \alpha_6 = \frac{\kappa^2}{2\nu^4}, \quad \alpha_5 = \frac{\kappa^2 \mu}{2\nu^2}$$

Where in it C_{ij} is the *Cotton* tensor which is defined as,

$$C^{ij} = \epsilon^{kl(i} \nabla_k R_l^{j)}$$

Following [53] we can write the action as

$$S = \int dt dx^3 (\mathcal{L}_0 + \mathcal{L}_1) \quad (7)$$

$$\mathcal{L}_0 = \sqrt{g} N \left(\frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_w R - 3\Lambda_w^2)}{8(1 - 3\lambda)} \right) \quad (8)$$

$$\mathcal{L}_1 = \sqrt{g} N \left(\frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2w^4} (C^{ij} - \frac{\mu w^2}{2} R^{ij}) (C_{ij} - \frac{\mu w^2}{2} R_{ij}) \right) \quad (9)$$

A curious case in the Hořava-Lifshitz theory is that the new mode satisfies a first order (in time derivatives) equation of motion. In linear approximation this extra freedom degree manifested only around non-static spatially inhomogeneous backgrounds. Blas. et. al [109] modified HL theory because of the following serious problems associated with this mode.

- The mode develops very fast exponential instabilities at short distances.
- It becomes strongly coupled at an extremely low cutoff scale

The significant result is that, so far it is proven that Hořava-Lifshitz theory has a vacuum solution as Lifshitz metric. So, it enjoys holography principle in non relativistic regime [108].

3 Modified $F(R)$ Hořava-Lifshitz Gravity

The action for standard $F(R)$ gravity can be written as[?]

$$S = \int d^4x \sqrt{g^{(3)}} N F(R). \quad (10)$$

If we take $z = 1$, then GR is recovered. We can rewrite the action as follows:

$$S = \frac{1}{2\kappa^2} \int dt d^3x \sqrt{g^{(3)}} N F(\tilde{R}), \quad \tilde{R} = K_{ij} K^{ij} - \lambda K^2$$

$$+ R^{(3)} + 2\mu \nabla_\mu (n^\mu \nabla_\nu n^\nu - n^\nu \nabla_\nu n^\mu) - L^{(3)}(g_{ij}^{(3)}), \quad (11)$$

where κ is the dimensionless gravitational coupling, μ and λ are the new constants which account for the violation of full diffeomorphism transformations. Note that the third term in the expression for \tilde{R} in the original Horava gravity theory can be omitted, as it becomes a total derivative. The term $L^{(3)}(g_{ij}^{(3)})$ is written as

$$L^{(3)}(g_{ij}^{(3)}) = E^{ij} G_{ijkl} E^{kl}, \quad (12)$$

where G_{ijkl} is the inverse of the generalized De Witt metric is,

$$G^{ijkl} = \frac{1}{2}(g^{(3)ik} g^{(3)jl} + g^{(3)il} g^{(3)jk}) - \lambda g^{(3)ij} g^{(3)kl}. \quad (13)$$

So, we have

$$G_{ijkl} = \frac{1}{2}(g_{ik}^{(3)} g_{jl}^{(3)} + g_{il}^{(3)} g_{jk}^{(3)}) - \bar{\lambda} g_{ij}^{(3)} g_{kl}^{(3)}, \bar{\lambda} = \frac{\lambda}{3\lambda - 1}. \quad (14)$$

Here it is important to note that G^{ijkl} is singular for $\lambda = 1/3$ and G_{ijkl} exist if $\lambda \neq 1/3$.

The expression for E_{ij} is constructed to satisfy the "detailed balance principle" [?] and defined as

$$\sqrt{g^{(3)}} E^{ij} = \frac{\delta W[g_{kl}^{(3)}]}{\delta g_{ij}^{(3)}}, \quad (15)$$

where the form of $W[g_{kl}^{(3)}]$ is given [?] for $z = 2$ and $z = 3$.

4 Cosmology of $F(R)$ Hořava-Lifshitz theory

We want to study of cosmological solutions for the theory described by action (11). The spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric is assumed as

$$ds^2 = -N^2 dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2. \quad (16)$$

We can see that, N can be taken to be just time-dependent in projectability condition and can be fixed to be unity, $N = 1$ by using the foliation-preserving diffeomorphisms (??). N depends on both time and spatial coordinates for the non-projectability condition. So, the assumption of the solution N is taken as unity.

For the metric (16), the scalar \tilde{R} is given by

$$\tilde{R} = \frac{3(1 - 3\lambda + 6\mu)H^2}{N^2} + \frac{6\mu}{N} \frac{d}{dt} \left(\frac{H}{N} \right). \quad (17)$$

For the action (11), and assuming the FRW metric (17), the second FRW equation can be obtained by varying the action with respect to the spatial metric $g_{ij}^{(3)}$, we get

$$\begin{aligned} 0 &= F(\tilde{R}) - 2(1 - 3\lambda + 3\mu)(\dot{H} + 3H^2)F'(\tilde{R}) - 2(1 - 3\lambda) \\ &\times \dot{\tilde{R}}F''(\tilde{R}) + 2\mu(\dot{\tilde{R}}^2 F^{(3)}(\tilde{R}) + \ddot{\tilde{R}}F''(\tilde{R})) - \kappa^2 \rho_m, \end{aligned} \quad (18)$$

where $\kappa^2 = 16\pi G$, is the pressure of perfect fluid that fills the universe, and $N = 1$. Note that, this equation becomes the usual second FLRW equation for convenient $F(\tilde{R})$ gravity (10) and the constants λ, μ can be taken as $\lambda = \mu = 1$. If we take the projectability condition, then the variation over N of the action (11) can be written as global constraint.

$$0 = \int d^3x [F(\tilde{R}) - 6(1 - 3\lambda + 3\mu)H^2 - 6\mu\dot{H} + 6\mu H\dot{\tilde{R}}F''(\tilde{R}) - \kappa^2 \rho_m]. \quad (19)$$

By using the ordinary conservation equation for the matter fluid $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$ and by integrating Eq. (14), we have

$$\begin{aligned} 0 &= F(\tilde{R}) - 6[(1 - 3\lambda + 3\mu)H^2 - \mu\dot{H}]F'(\tilde{R}) + 6\mu H\dot{\tilde{R}} \\ &\times F''(\tilde{R}) - \kappa^2 \rho_m - \frac{C}{a^3}, \end{aligned} \quad (20)$$

where C is the integrating constant, taken to be zero, according to the constraint equation (19). On the other hand, if we take the non-projectability condition, we can obtain the equation (20) directly which corresponds to the first FLRW equation, by variation over N .

The scalar curvature (17) can be written as

$$\tilde{R} = 3(1 - 3\lambda + 6\mu)H^2 + 6\mu HH'. \quad (21)$$

5 Non-Equilibrium Description of Thermodynamics in $F(R)$ Hořava-Lifshitz Gravity

5.1 Energy Density and Pressure of Dark Components

The energy density and dark components can be evaluated by field equations. So we can rewrite the FLRW Eqs. (18) and (20) in $F(R)$ Hořava-Lifshitz gravity as

$$H^2 = \frac{\kappa^2}{3(3\lambda - 1)}\rho_{eff}, \quad \dot{H} = \frac{-\kappa^2}{2(3\lambda - 1)}(\rho_{eff} + p_{eff}), \quad (22)$$

where $H = \dot{a}/a$ is the Hubble parameter and dot denotes the derivative w.r.t 't'.

$$\rho_{eff} = \hat{\rho}_{de} + \rho_m. \quad (23)$$

$$P_{eff} = \hat{p}_{de} + p_m. \quad (24)$$

here $\hat{\rho}_{de}$ and \hat{P}_{de} are the energy density and pressure of dark components are given by

$$\begin{aligned}\hat{\rho}_{de} &= \frac{1}{\kappa^2}[-f(\tilde{R}) + 3(1 - 3\lambda + 6\mu)H^2F(\tilde{R}) + 6\mu\dot{H}F(\tilde{R}) \\ &\quad - 6\mu H\dot{\tilde{R}}F'(\tilde{R})].\end{aligned}\quad (25)$$

$$\begin{aligned}\hat{p}_{de} &= \frac{1}{\kappa^2}[f(\tilde{R}) - 6\mu\dot{H}F(\tilde{R}) - 3(1 - 3\lambda + 6\mu)H^2F(\tilde{R}) - 2 \\ &\quad \times (1 - 3\lambda)H\dot{\tilde{R}}F'(\tilde{R}) + 2\mu\dot{\tilde{R}}^2F''(\tilde{R}) + 2\mu\ddot{\tilde{R}}F'(\tilde{R})].\end{aligned}\quad (26)$$

Leading to

$$\dot{\hat{\rho}}_{de} + 3H(\hat{\rho}_{de} + \hat{p}_{de}) = -3(1 - 3\lambda)H^2\dot{\tilde{R}}. \quad (27)$$

$$\dot{\rho}_m + 3H\rho_m = 0. \quad (28)$$

where, by hat we label all the quantities in the non-equilibrium description of thermodynamics. It is easy to check that the standard continuity equation does not hold due to $\dot{\tilde{R}} \neq 0$ in Eq. (27), and dot denotes the derivative with respect to 't' and prime denote the derivative with respect to \tilde{R} .

5.2 First Law of Thermodynamics

In $F(R)$ Hořava-Lifshitz gravity by using the relation $h^{\alpha\beta}\partial_\alpha\tilde{r}\partial_\beta\tilde{r} = 0$ we determines the dynamical apparent horizon. In the flat FLRW spacetime, the radius \tilde{r}_A of the apparent horizon is,

$$\tilde{r}_A = \frac{1}{H}, \quad (29)$$

the time derivative of Eq. (29) is

$$-\frac{d\tilde{r}_A}{\tilde{r}_A^3} = \frac{\dot{H}}{H}dt. \quad (30)$$

Substituting Eq. (22) into Eq. (30) we get,

$$\frac{F}{4\pi G}d\tilde{r}_A = \frac{\tilde{r}_A^3 H}{3\lambda - 1}(\hat{\rho}_t + \hat{p}_t)dt. \quad (31)$$

Where $\hat{\rho}_t \equiv \hat{\rho}_{de} + \rho_m$, $\hat{p}_t \equiv \hat{p}_{de} + p_m$ are the total energy density and pressure of the universe respectively.

The Bekenstein-Hawking horizon (killing) entropy is defined as $S = A/4G$, where $A = 4\pi\tilde{r}_A^2$ is the area of the apparent horizon [?]. In modified $F(\tilde{R})$ gravity, a horizon entropy \hat{S} associated with the Wald entropy \hat{S} is a Neother charge, is defined as $\hat{S} = A/4G_{eff}$, where $G_{eff} = G/f'$ with $f' = df(\tilde{R})/d(\tilde{R})$ is the effective gravitational coupling in $F(\tilde{R})$ gravity. It is remarkable to mention here that the Wald entropy \hat{S} in $F(\tilde{R})$ gravity kept the same form in both formalisms of metric and palatini.

The entropy of black holes in $F(\tilde{R})$ gravity is

$$\hat{S} = \frac{FA}{4G}. \quad (32)$$

By using Eqs. (31) and (32), we get

$$\frac{1}{2\pi\tilde{r}_A}d\hat{S} = \frac{4\pi}{3\lambda-1}\tilde{r}_A^3 H(\hat{\rho}_t + \hat{p}_t)dt + \frac{\tilde{r}_A}{2G}dF. \quad (33)$$

The associated temperature of the apparent horizon has the following Hawking temperature T_H

$$T_H = \frac{|\kappa_{sg}|}{2\pi}. \quad (34)$$

Where κ_{sg} is the surface gravity

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_0(\sqrt{-h}h^{\alpha\beta}\partial_\beta\tilde{r}) \quad (35)$$

$$\begin{aligned} \kappa_{sg} &= -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) = -\frac{\tilde{r}_A}{2}(2H^2 + \dot{H}) \\ &= -\frac{2\pi G}{3F(3\lambda-1)}(\hat{\rho}_t - 3\hat{p}_t). \end{aligned} \quad (36)$$

Where $h = \det(h_{\alpha\beta})$. From Eq.(36), we see that $\kappa_{sg} \leq 0$ if the total equation of state (EoS) $\omega_t \equiv \hat{P}_t/\hat{\rho}_t$ satisfies $\omega_t \leq 1/3$.

By solving the Eqs. (34) and (36), we have

$$T_H = \frac{1}{2\pi\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right). \quad (37)$$

By multiplying the term $(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})$ for Eq. (33), we have

$$T_H d\hat{S} = \frac{4\pi}{3\lambda-1}\tilde{r}_A^3 H(\hat{\rho}_t + \hat{p}_t)dt - \frac{2\pi}{3\lambda-1}\tilde{r}_A^2(\hat{\rho}_t + \hat{p}_t)d\tilde{r}_A + \frac{T_H}{G}\pi\tilde{r}_A^2 dF. \quad (38)$$

The Misner Sharp energy E in general relativity is defined as $E \equiv \tilde{r}_A/2G$. Since $G_{eff} = G/F$ in $F(\tilde{R})$ gravity, may be written as

$$\hat{E} = \frac{\tilde{r}_A F}{2G}, \quad (39)$$

by combining Eqs. (30) and (39), we get

$$\hat{E} = \frac{3FH^2}{8\pi G}V = \frac{1}{3\lambda-1}V\hat{\rho}_t. \quad (40)$$

Where $V = 4\pi\tilde{r}_A^3/3$ is the volume inside the apparent horizon. It shows that from the Eq. (40) \hat{E} corresponds to the total intrinsic energy. It is also clear

that from this Eq. (40) that $F \geq 0$ so that $\hat{E} \geq 0$. The effective gravitational coupling in $F(\tilde{R})$ gravity becomes positive (no ghost) . .

Using Eqs. (27) and (28), we have

$$d\hat{E} = -\frac{4\pi}{3\lambda-1}\tilde{r}_A^3 H(\hat{\rho}_t + \hat{p}_t)dt + \frac{4\pi}{3\lambda-1}\tilde{r}_A^2 \hat{\rho}_t d\tilde{r}_A + \frac{\tilde{r}_A}{2G}dF. \quad (41)$$

By using the Eqs. (38) and (41)

$$T_H d\hat{S} = -d\hat{E} + \frac{2\pi}{3\lambda-1}\tilde{r}_A^2 (\hat{\rho}_t - \hat{p}_t)d\tilde{r}_A + \frac{\tilde{r}_A}{2G}(1 + 2\pi\tilde{r}_A T_H)dF. \quad (42)$$

By introducing the work density [110],

$$\hat{W} \equiv -\frac{1}{2}(T^{(M)\alpha\beta}h_{\alpha\beta} + \hat{T}^{(DE)\alpha\beta}h_{\alpha\beta}), \quad (43)$$

$$= -\frac{1}{2}(\hat{\rho}_t - \hat{p}_t). \quad (44)$$

With $\hat{T}^{(de)\alpha\beta}$ being the energy-momentum tensor of the dark components, Eq. (43) is rewritten as

$$T_H d\hat{S} = -d\hat{E} + \frac{1}{3\lambda-1}\hat{W}dV + \frac{\tilde{r}_A}{2G}(1 + 2\pi\tilde{r}_A T_H)dF. \quad (45)$$

Which can be described as

$$T_H d\hat{S} + T_H d_i \hat{S} = -d\hat{E} + \frac{1}{3\lambda-1}\hat{W}dV. \quad (46)$$

Where

$$\begin{aligned} d_i \hat{S} &= -\frac{1}{T_H} \frac{\tilde{r}_A}{2G}(1 + 2\pi\tilde{r}_A T_H)dF = -\left(\frac{\hat{E}}{T_H} + \hat{S}\right)\frac{dF}{F} \\ &= -\frac{\pi}{GH^2} \frac{4H^2 + \dot{H}}{2H^2 + \dot{H}}dF. \end{aligned} \quad (47)$$

The term $d_i \hat{S}$ is a additional term which can be interpreted as an entropy production term in the non-equilibrium thermodynamics.

5.3 Second Law of Thermodynamics

Recently, the second law of thermodynamics has been studied in the context of modified $F(R)$ Hořava-Lifshitz gravitational theory. It may be interesting to investigate its validity in $F(\tilde{R})$ gravity. For this purpose, we have to show that

$$\Xi \equiv \frac{d\hat{S}}{dt} + \frac{d_i \hat{S}}{dt} + \frac{d\hat{S}_t}{dt} \geq 0. \quad (48)$$

Where \hat{S} is the horizon entropy in $F(\tilde{R})$ gravity and \hat{S}_{tot} is the entropy due to all the matter and energy sources inside the horizon. The Gibbs equation including all matter and energy fluid is given by

$$T_H dS_t = d(\rho_t V) + p_t dV = V d\rho_t + (\rho_t + p_t) dV. \quad (49)$$

Where T_H and \hat{S}_t denotes the temperature and entropy of total energy inside the horizon, respectively. The main assumption is that here we suppose that inside and outside of the apparent horizon remain in thermal equilibrium with the same temperature.

By using the Eqs. (22), (46) and (49), we obtain,

$$\Xi = \frac{F}{2G} \frac{\dot{H}^2}{H^4}. \quad (50)$$

$$J = 144H^2 \dot{H}^2 F \geq 0. \quad (51)$$

Which is always met because $F > 0$ and $\hat{E} > 0$. Hence the second law of thermodynamics can be satisfied in $F(R)$ Hořava-Lifshitz gravity. We can see that from Eq. (51) $J \geq 0$ irrespective of the sign of \dot{H} .

We conclude that we have used the Physical temperature as the temperature of apparent horizon. This temperature clearly depends on the energy momentum tensor of the dark components of $F(R)$ Hořava-Lifshitz gravity. The temperature of matter species in a cosmological setup is determined in a standard way. We have concentrated in our discussions (second law of thermodynamics in $F(R)$ Hořava-Lifshitz gravity) on the case in which temperature of the universe inside the horizon is equal to that of the apparent horizon.

6 Equilibrium Description of Thermodynamics in $F(R)$ Hořava-Lifshitz Gravity

In the case of non-equilibrium description of thermodynamics the entropy production term $d_i \hat{S}$, the R.H.S of Eq. (27) does not vanish and the equation of continuity for $\hat{\rho}_{de}$ and \hat{P}_{de} does not hold for this purpose. We demonstrated that in the case of equilibrium description of thermodynamics by redefining the energy density and pressure of dark components to meet the continuity equation. So, there can be no extra entropy production term in the equilibrium description in $F(R)$ Hořava-Lifshitz gravity.

6.1 Energy Density and Pressure of Dark Components

The Friedmann equations are for equilibrium description in $F(R)$ Hořava-Lifshitz gravity

$$H^2 = \frac{\kappa^2}{3(3\lambda - 1)} \rho_{eff}, \quad \dot{H} = \frac{-\kappa^2}{2(3\lambda - 1)} (\rho_{eff} + p_{eff}), \quad (52)$$

$$\rho_{eff} = \rho_{de} + \rho_m. \quad (53)$$

$$p_{eff} = p_{de} + p_m. \quad (54)$$

The energy density and pressure of dark components can be rewritten as

$$\begin{aligned} \rho_{de} &= -f(\tilde{R}) + 6(1 - 3\lambda + 3\mu)H^2 F(\tilde{R}) + 6\mu\dot{H}F(\tilde{R}) \\ &\times \mu H \frac{dF(\tilde{R})}{dt}. \end{aligned} \quad (55)$$

$$\begin{aligned} p_{de} &= f(\tilde{R}) - 2(1 - 3\lambda + 3\mu)(3H^2 + \dot{H})F(\tilde{R}) - 2 \\ &\times (1 - 3\lambda)H \frac{dF(\tilde{R})}{dt} + 2\mu \frac{d^2 F(\tilde{R})}{dt^2}. \end{aligned} \quad (56)$$

Which clearly satisfy the standard equations of continuity, i.e.,

$$\dot{\rho}_{de} + 3H(\rho_{de} + p_{de}) = 0. \quad (57)$$

6.2 First law of thermodynamics

By using Eq. (52) and (31), we get

$$\frac{1}{4\pi G} d\tilde{r}_A = \frac{r_A^3 H}{3\lambda - 1} (\rho_t + p_t) dt. \quad (58)$$

Where $\rho_t = \rho_{de} + \rho_m$, $p_t = p_{de} + p_m$, by introducing the horizon entropy $S = A/4G$ and using Eq. (58), we have

$$\frac{1}{2\pi\tilde{r}_A} dS = \frac{4\pi}{3\lambda - 1} \tilde{r}_A^3 H (\rho_t + p_t) dt. \quad (59)$$

From the horizon temperature in Eq. (37) and (59), we get,

$$T_H dS = \frac{4\pi}{3\lambda - 1} \tilde{r}_A^3 H (\rho_t + p_t) dt - \frac{2\pi}{3\lambda - 1} \tilde{r}_A^2 (\rho_t + p_t) d\tilde{r}_A. \quad (60)$$

By defining the Misner-sharp energy as $E = \frac{\tilde{r}_A}{2G}$. By solving the Eqs. we have,

$$E = \frac{3FH^2}{8\pi G} V = \frac{1}{3\lambda - 1} V \rho_t. \quad (61)$$

We get

$$dE = -\frac{4\pi}{3\lambda - 1} \tilde{r}_A^3 H (\rho_t + p_t) dt + \frac{4\pi}{3\lambda - 1} \tilde{r}_A^2 \rho_t d\tilde{r}_A. \quad (62)$$

It is noted that, there does not exists any additional term proportional to the dF on the R.H.S due to the continuity equation Eq (58). From Eqs. (60) and (62), we get the following equation corresponding to the first law of thermodynamics.

$$T_H dS = -dE + \frac{1}{3\lambda - 1} W dV. \quad (63)$$

Where the work density is given by

$$W = \frac{1}{2}(\rho_t - p_t). \quad (64)$$

So, by redefining ρ_{DE} and P_{DE} , the equation of continuity can be satisfies, we can realize the existence of the equilibrium thermodynamical phase in $f(\tilde{R})$ gravity.

by using the Eqs. (52), (59), and (57), we get

$$\dot{S} = -\frac{2\pi}{G} \frac{\dot{H}}{H^3}. \quad (65)$$

Since $\dot{S} \propto -\dot{H}/H^3$, the horizon entropy increases in the expanding universe as long as the null energy condition $\rho_t + P_t \geq 0$ is satisfied, in which $\dot{H} \leq 0$.

There are two main reasons why we can obtain the equilibrium description of thermodynamics:

- First of all, the Bekenstein-Hawking area entropy is valid here just by a formal redefinition of the effective G_{eff} .
- We satisfy continuity equation by redefining of the effective energy density and pressure of dark components .

The basic relation between the horizon entropy S in the equilibrium description and \hat{S} in the non-equilibrium description are given as follows:

$$dS = d\hat{S} + d_i\hat{S} + \frac{\tilde{r}_A}{2GT_H}dF - \frac{2\pi(1-F)}{G} \frac{\dot{H}}{H^3}dt. \quad (66)$$

By using the relation (47), (58) and (66), we have,

$$dS = \frac{1}{F}d\hat{S} + \frac{1}{F} \frac{2H^2 + \dot{H}}{4H^2 + \dot{H}} d_i\hat{S}. \quad (67)$$

Where $d_i\hat{S}$ is given by Eq. (47). Because of $dF \neq 0$, we obtain a non zero difference between S and \hat{S} .

6.3 Second Law of Thermodynamics

In the case of the equilibrium description, to evaluate the second law of thermodynamics, we write the Gibbs equation in terms of all matter and energy fluid as

$$T_H dS_t = d(\rho_t V) + p_t dV = V d\rho_t + (\rho_t + p_t) dV. \quad (68)$$

The second law of thermodynamics argues that the total entropy of the system never decreases in time:

$$\frac{dS_{sum}}{dt} \equiv \frac{dS}{dt} + \frac{dS_t}{dt} \geq 0. \quad (69)$$

Where $S_{sum} = S + S_t$. Consequently, we have

$$\frac{dS_{sum}}{dt} = \frac{2\pi}{G} \frac{\dot{H}^2}{H^2} \frac{1}{H(2H^2 + \dot{H})}. \quad (70)$$

By using $V = 4\pi\tilde{r}_A^3/3$, and Eqs. (37)(52) and (65). Hence the relation with Eq. (69) leads to the condition

$$Y \equiv 12H(2H^2 + \dot{H}) \geq 0. \quad (71)$$

In the flat FLRW expanding background ($H \geq 0$), the second law of thermodynamics can be satisfied in $F(R)$ Hořava-Lifshitz gravity, $R = 6(2H^2 + \dot{H})$ for the flat FLRW space-time and the condition in Eq.(71) clearly holds as $\tilde{R} \geq 0$.

7 Summary and conclusion

Thermodynamic laws in the non relativistic regime of $f(R)$ gravity, $F(R)$ Hořava-Lifshitz gravity investigated in both equilibrium and non equilibrium modes. By assuming that the inside and outside of apparent horizon are in thermal equilibrium, we proved that the second law of thermodynamics can be satisfied for both cases. In the non-equilibrium framework, it has been shown that the second law of the thermodynamics can be satisfied regardless of the sign of the time derivative of the Hubble parameter and in the equilibrium framework, the second law of thermodynamics can be shown by analogy with the same non-negative quantity which is related to the scalar curvature in (GR) is positive or equal to zero in cosmology.

Finally, we can conclude that our result of second law of thermodynamics in $F(R)$ Hořava-Lifshitz gravity is non-trivial.

References

- [1] S. Perlmutter et al. [SNCP Collaboration], *Astrophys. J.* 517, 565 (1999); A. G. Riess et al.[SNST Collaboration], *Astron. J.* 116, 1009 (1998); D. N. Spergel et al. [WMAP Collaboration], *Astrophys. J. Suppl.* 148, 175 (2003); *ibid.* 170, 377 (2007); E. Komatsu et al. [WMAP Collaboration], *ibid.* 180, 330 (2009); E. Komatsu et al. [WMAP Collaboration], *Astrophys. J. Suppl.* 192, 18 (2011); M. Tegmark et al., *Phys. Rev. D* 69, 103501 (2004); U. Seljak et al. [SDSS Collaboration], *Phys. Rev. D* 71, 103515 (2005); D. J. Eisenstein et al., *Astrophys. J.* 633, 560 (2005); B. Jain and A. Taylor, *Phys. Rev. Lett.* 91, 141302 (2003).
- [2] P. J. E. Peebles and B. Ratra, *Rev. Mod. Phys.* **75**, 559 (2003); T. Padmanabhan, *Phys. Repts.* **380**, 235 (2003).
- [3] H. A. Buchdahl, *Mon. Not. Roy. Astron. Soc.* 150, 1 (1970).

- [4] S. Nojiri and S. D. Odintsov, Phys. Rept. **505**, 59 (2011) [arXiv:1011.0544 [gr-qc]].
- [5] S. Nojiri and S. D. Odintsov, eConf C **0602061** (2006) 06 [Int. J. Geom. Meth. Mod. Phys. **4** (2007) 115] doi:10.1142/S0219887807001928 [hep-th/0601213].
- [6] A. De Felice and S. Tsujikawa, Living Rev. Rel. **13**, 3 (2010) [arXiv:1002.4928 [gr-qc]].
- [7] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).
- [8] T. Koivisto, Phys. Rev. D **76**, 043527 (2007); A. A. Starobinsky, JETP Lett. **86**, 157 (2007); B. Li, J. D. Barrow, and D. F. Mota, Phys. Rev. D **76**, 044027 (2007); A. Azadi, D. Momeni and M. Nouri-Zonoz, Phys. Lett. B **670** (2008) 210, [arXiv:0810.4673 [gr-qc]]; V. Faraoni, Phys. Rev. D **72**, 061501 (2005); V. Faraoni and S. Nadeau, Phys. Rev. D **72**, 124005 (2005); A. Abebe, D. Momeni and R. Myrzakulov, arXiv:1507.03265 [gr-qc]; D. Momeni, H. Gholizade, M. Raza and R. Myrzakulov, Int. J. Mod. Phys. A **30** (2015) 16, 1550093, [arXiv:1502.05000 [gr-qc]]; M. U. Farooq, M. Jamil, D. Momeni and R. Myrzakulov, Can. J. Phys. **91**, 703 (2013), [arXiv:1306.1637 [astro-ph.CO]]; M. Jamil, F. M. Mahomed and D. Momeni, Phys. Lett. B **702** (2011) 315, [arXiv:1105.2610 [physics.gen-ph]]; S. Carloni, P. K. S. Dunsby, and A. Troisi, Phys. Rev. D **77**, 024024 (2008); S. Nojiri and S. D. Odintsov, Phys. Lett. B **652**, 343 (2007); S. Tsujikawa, Phys. Rev. D **77**, 023507 (2008).
- [9] D. Momeni and R. Myrzakulov, Astrophys. Space Sci. **360**, no. 1, 28 (2015), [arXiv:1511.01205 [physics.gen-ph]].
- [10] T. Chiba, Phys. Lett. B **575**, 1 (2003); A. L. Erickcek, T. L. Smith, and M. Kamionkowski, Phys. Rev. D **74**, 121501 (2006); T. Chiba, T. L. Smith, and A. L. Erickcek, Phys. Rev. D **75**, 124014 (2007); S. Nojiri and S. D. Odintsov, Phys. Lett. B **659**, 821 (2008); S. Capozziello, A. Stabile, and A. Troisi, Phys. Rev. D **76**, 104019 (2007); S. Capozziello, A. Stabile, and A. Troisi, Class. Quantum Grav. **25**, 085004 (2008).
- [11] G. J. Olmo, Phys. Rev. D **75**, 023511 (2007).
- [12] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
- [13] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003); Gen. Rel. Grav. **36** (2004) 1765; V. Faraoni, Phys. Rev. D **74**, 023529 (2006); T. Faulkner, M. Tegmark, E. F. Bunn, and Y. Mao, Phys. Rev. D **76**, 063505 (2007); M. Khurshudyan, N. S. Mazhari, D. Momeni, R. Myrzakulov and M. Raza, Int. J. Theor. Phys. **54**, no. 2, 484 (2015) [arXiv:1403.0081 [gr-qc]]; C. S. J. Pun, Z. Kovacs, and T. Harko, Phys. Rev. D **78**, 024043 (2008).

- [14] I. Sawicki and W. Hu, Phys. Rev. D **75**, 127502 (2007).
- [15] L. Amendola and S. Tsujikawa, Phys. Lett. B **660**, 125 (2008).
- [16] S. Nojiri and S. D. Odintsov, Phys. Lett. B **657**, 238 (2007); S. Nojiri and S. D. Odintsov, Phys. Rev. D **77**, 026007 (2008).
- [17] S. Capozziello, V. F. Cardone, and A. Troisi, JCAP **0608**, 001 (2006); S. Capozziello, V. F. Cardone, and A. Troisi, Mon. Not. R. Astron. Soc. **375**, 1423 (2007).
- [18] A. Borowiec, W. Godlowski, and M. Szydowski, Int. J. Geom. Meth. Mod. Phys. **4** (2007) 183
- [19] C. F. Martins and P. Salucci, Mon. Not. R. Astron. Soc. **381**, 1103 (2007).
- [20] C. G. Boehmer, T. Harko, and F. S. N. Lobo, Astropart. Phys. **29**, 386 (2008).
- [21] C. G. Boehmer, T. Harko, and F. S. N. Lobo, JCAP **03**, 024 (2008).
- [22] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010); F. S. N. Lobo, arXiv:0807.1640 [gr-qc]. S. Capozziello and V. Faraoni, “Beyond Einstein Gravity”, Springer, 2010.
- [23] O. Bertolami, C. G. Boehmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007).
- [24] T. Harko, Phys. Lett. B **669**, 376 (2008).
- [25] S. Nojiri and S. D. Odintsov, Phys. Lett. B **599**, 137 (2004); V. Faraoni, Phys. Rev. D **76**, 127501 (2007); T. P. Sotiriou, Phys. Lett. B **664**, 225 (2008); O. Bertolami, T. Harko, F. S. N. Lobo, and J. Paramos, arXiv:0811.2876 (2008); O. Bertolami and M. C. Sequeira, Phys. Rev. D **79**, 104010 (2009); S. Nesseris, Phys. Rev. D **79**, 044015 (2009); T. Harko, Phys. Rev. D **81**, 084050 (2010); S. Thakur, A. A. Sen, and T. R. Seshadri, arXiv:1007.5250 (2010).
- [26] T. Harko, Phys. Rev. D **81**, 044021 (2010).
- [27] T. Harko, T. S. Koivisto and F. S. N. Lobo, arXiv:1007.4415 (2010).
- [28] T. Harko and F. S. N. Lobo, Eur. Phys. J. C **70**, 373 (2010).
- [29] N. J. Poplawski, arXiv:gr-qc/0608031.
- [30] P. Horava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].
- [31] P. Horava, arXiv:0811.2217 [hep-th].
- [32] P. Horava, JHEP **0903**, 020 (2009) [arXiv:0812.4287 [hep-th]].

- [33] P. Hořava, arXiv:0902.3657 [hep-th].
- [34] G. E. Volovik, arXiv:0904.4113 [gr-qc].
- [35] D. Orlando and S. Reffert, arXiv:0905.0301 [hep-th].
- [36] T. Nishioka, arXiv:0905.0473 [hep-th].
- [37] R. A. Konoplya, arXiv:0905.1523 [hep-th].
- [38] C. Charmousis, G. Niz, A. Padilla and P. M. Saffin, arXiv:0905.2579 [hep-th].
- [39] T. P. Sotiriou, M. Visser and S. Weinfurtner, arXiv:0905.2798 [hep-th].
- [40] C. Bogdanos and E. N. Saridakis, arXiv:0907.1636 [hep-th].
- [41] J. Kluson, arXiv:0907.3566 [hep-th].
- [42] M. Li and Y. Pang, arXiv:0905.2751 [hep-th].
- [43] M. Visser, arXiv:0902.0590 [hep-th].
- [44] J. Chen and Y. Wang, arXiv:0905.2786 [gr-qc].
- [45] B. Chen and Q. G. Huang, arXiv:0904.4565 [hep-th].
- [46] F. W. Shu and Y. S. Wu, arXiv:0906.1645 [hep-th].
- [47] J. Z. Tang and B. Chen, arXiv:0909.4127 [hep-th].
- [48] G. Calcagni, arXiv:0904.0829 [hep-th].
- [49] E. Kiritsis and G. Kofinas, arXiv:0904.1334 [hep-th].
- [50] S. Nojiri, S. D. Odintsov, arXiv:0905.4213 [hep-th].
- [51] G. Leon, E. N. Saridakis, JCAP 0911: 006, (2009).
- [52] S. Dutta, E. N. Saridakis, arXiv:0911.1435 [hep-th].
- [53] H. Lu, J. Mei and C. N. Pope, arXiv:0904.1595 [hep-th].
- [54] H. Nastase, arXiv:0904.3604 [hep-th].
- [55] M. Minamitsuji, arXiv:0905.3892 [astro-ph.CO].
- [56] Y. -F. Cai, X. Zhang, Phys. Rev.D **80**, 043520, (2009)
- [57] X. Gao, arXiv:0904.4187 [hep-th].
- [58] B. Chen, S. Pi and J. Z. Tang, arXiv:0905.2300 [hep-th].
- [59] X. Gao, Y. Wang, R. Brandenberger and A. Riotto, arXiv:0905.3821 [hep-th].

- [60] A. Wang and R. Maartens, arXiv:0907.1748 [hep-th].
- [61] T. Kobayashi, Y. Urakawa and M. Yamaguchi, arXiv:0908.1005 [astro-ph.CO].
- [62] C. Ding, S. Chen, J. Jing, arXiv:0909.2490v1 [gr-qc].
- [63] T. Takahashi and J. Soda, arXiv:0904.0554 [hep-th].
- [64] S. Koh, arXiv:0907.0850 [hep-th].
- [65] R. Brandenberger, arXiv:0904.2835 [hep-th].
- [66] R. H. Brandenberger, arXiv:0905.1514 [hep-th].
- [67] Y. F. Cai and E. N. Saridakis, arXiv:0906.1789 [hep-th].
- [68] U. H. Danielsson and L. Thorlacius, JHEP **0903**, 070 (2009) [arXiv:0812.5088 [hep-th]].
- [69] A. Kehagias and K. Sfetsos, arXiv:0905.0477 [hep-th];
- [70] R. B. Mann, arXiv:0905.1136 [hep-th].
- [71] G. Bertoldi, B. A. Burrington and A. Peet, arXiv:0905.3183 [hep-th].
- [72] A. Castillo and A. Larranaga, arXiv:0906.4380 [gr-qc].
- [73] M. Botta-Cantcheff, N. Grandi and M. Sturla, arXiv:0906.0582 [hep-th].
- [74] R. G. Cai, L. M. Cao and N. Ohta, Phys. Rev. D **80**, 024003, (2009).
- [75] D. Momeni, arXiv:0910.0594v2 [gr-qc].
- [76] E. N. Saridakis, arXiv:0905.3532 [hep-th].
- [77] A. Wang and Y. Wu, arXiv:0905.4117 [hep-th].
- [78] C. Appignani, R. Casadio and S. Shankaranarayanan, arXiv:0907.3121 [hep-th].
- [79] M. R. Setare, arXiv:0909.0456 [hep-th].
- [80] S. S. Kim, T. Kim and Y. Kim, arXiv:0907.3093 [hep-th].
- [81] T. Harko, Z. Kovacs and F. S. N. Lobo, arXiv:0908.2874 [gr-qc].
- [82] M. R. Setare and D. Momeni, Int. J. Mod. Phys. D **19**, 2079 (2010) doi:10.1142/S0218271810018232 [arXiv:0911.1877 [hep-th]].
- [83] M. R. Setare and D. Momeni, Int. J. Theor. Phys. **50** (2011) 106 doi:10.1007/s10773-010-0498-8 [arXiv:1001.3767 [physics.gen-ph]].

- [84] M. R. Setare and D. Momeni, *Int. J. Theor. Phys.* **51**, 198 (2012) doi:10.1007/s10773-011-0894-8 [arXiv:1009.0918 [hep-th]].
- [85] D. Momeni, *Int. J. Theor. Phys.* **50**, 1493 (2011) doi:10.1007/s10773-010-0659-9 [arXiv:0910.0594 [gr-qc]].
- [86] M. R. Setare and D. Momeni, *Mod. Phys. Lett. A* **26** (2011) 151 doi:10.1142/S0217732311034700 [arXiv:1002.0185 [hep-th]].
- [87] M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen and A. Tureanu, *Class. Quant. Grav.* **27** (2010) 185021 [Class. Quant. Grav. **29** (2012) 159501] doi:10.1088/0264-9381/29/15/159501, 10.1088/0264-9381/27/18/185021 [arXiv:1001.4102 [hep-th]].
- [88] J. Kluson, *JHEP* **0911**, 078 (2009) doi:10.1088/1126-6708/2009/11/078 [arXiv:0907.3566 [hep-th]].
- [89] N. Kiriushcheva, P. G. Komorowski and S. V. Kuzmin, arXiv:1112.5639 [gr-qc].
- [90] N. Kiriushcheva, P. G. Komorowski and S. V. Kuzmin, arXiv:1111.2647 [gr-qc].
- [91] A. J. Lopez-Revelles, R. Myrzakulov and D. Saez-Gomez, *Phys. Rev. D* **85**, 103521 (2012) doi:10.1103/PhysRevD.85.103521 [arXiv:1201.5647 [gr-qc]].
- [92] A. Jawad and S. Rani, *Astrophys. Space Sci.* **359**, no. 1, 23 (2015).
- [93] S. K. Chakrabarti, K. Dutta and A. A. Sen, *Phys. Lett. B* **711**, 147 (2012) doi:10.1016/j.physletb.2012.03.080 [arXiv:1108.2781 [astro-ph.CO]].
- [94] A. Jawad and S. Rani, *Astrophys. Space Sci.* **357**, no. 1, 88 (2015).
- [95] J. Kluson, *Phys. Rev. D* **84**, 104014 (2011) doi:10.1103/PhysRevD.84.104014 [arXiv:1107.5660 [hep-th]].
- [96] A. Jawad and S. Chattopadhyay, *Astrophys. Space Sci.* **357**, no. 1, 37 (2015).
- [97] D. Saez-Gomez, *J. Phys. Conf. Ser.* **314**, 012055 (2011) doi:10.1088/1742-6596/314/1/012055 [arXiv:1012.4605 [gr-qc]].
- [98] A. Bravetti, D. Momeni, R. Myrzakulov and H. Quevedo, *Gen. Rel. Grav.* **45**, 1603 (2013) doi:10.1007/s10714-013-1549-2 [arXiv:1211.7134 [gr-qc]].
- [99] A. Jawad, *Eur. Phys. J. Plus* **129**, no. 10, 207 (2014).
- [100] A. Jawad and S. Chattopadhyay, *Astrophys. Space Sci.* **353** (2014) 1, 293.
- [101] J. Bellorin, A. Restuccia and A. Sotomayor, *Phys. Rev. D* **90**, no. 4, 044009 (2014)

- [102] J. Kluson, S. Nojiri, S. D. Odintsov and D. Saez-Gomez, Eur. Phys. J. C **71**, 1690 (2011) doi:10.1140/epjc/s10052-011-1690-6 [arXiv:1012.0473 [hep-th]].
- [103] D. Saez-Gomez, Phys. Rev. D **83**, 064040 (2011) doi:10.1103/PhysRevD.83.064040 [arXiv:1011.2090 [hep-th]].
- [104] Misner, C.W., Deser, S., Arnowitt, R.L.: "Gravitation: an introduction to current research" Louis Willen ed. (Wiley 1962), chapter 7,.
- [105] Gao, C.: Phys. Lett.B **684**(2010)85.
- [106] Thorne, K.S. et al.: Gravitation (Freemann, San Francisco, 1973).
- [107] D. Blas, O. Pujolas, S. Sibiryakov, JHEP, 10, (2009), 029.
- [108] T. Griffin, P. Hoava and C. M. Melby-Thompson, Phys. Rev. Lett. **110**, no. 8, 081602 (2013) doi:10.1103/PhysRevLett.110.081602 [arXiv:1211.4872 [hep-th]].
- [109] Blas, D. et al.: Phys. Rev. Lett. **104**(2010)181302.
- [110] Bardeen, J.M. et al.: Commun. Math. Phys. **31**(1973)161; Bekenstein, J.D.: Phys. Rev. D **7**(1973)2333; Hawking, S.W.: Commun. Math. Phys. **43**(1975)199; Gibbon, G.W., Hawking, S.W.: Phys. Rev. D **15**(1977)2738.